Problem 1

1) (a^l×a^m)×a^n = a^l×(a^m×a^n), 构成半群

任意m∈Z有a^0×a^m = a^m×a^0 = a^m, 构成独异点

任意m∈Z有n=-m使a^m×a^n = 1, 构成群.

2) a, b, c∈Q+, (a×b)×c = a×(b×c), 构成半群

任意a∈Q+有a×1 = 1×a = a, 构成独异点

任意a∈Q+有b=1/a使a×b = 1, 构成群

3) a, b, c∈Q+, (a+b)+c = a+(b+c), 构成半群

任意a∈Q+有a+0 = 0+a = a, 但0∉Q+, 不构成独异点, 不构成群

4) 满足结合律, 构成半群; 有幺元0, 构成独异点; 多项式取负即为逆元, 构成群

5) 满足结合律, 构成半群; 有幺元1, 构成独异点

非常数多项式取倒数不是多项式, 多项式不一定存在逆元, 不构成群

6) x, y, z∈C且x^l=1, y^m=1, z^n=1, (xy)z = x(yz) = xyz

(xyz)^(lmn) = 1^(mn) × 1^(ln) × 1^(lm) = 1×1×1 = 1, 构成半群

任意x∈Un有1·x = x·1 = x, 构成独异点

对x=a+bi有y=(a-bi)/(a^2+b^2)使x·y = 1, 构成群

Problem 2

∀x,y∈S, x\*y=x∈S, \*在S上封闭, (S, \*)为代数系统

(x\*y)\*z = x\*z = x, x\*(y\*z) = x\*y = x, 满足结合性, S关于\*运算构成半群

Problem 3

V=<{a, b}, \*>是半群, 则\*在{a, b}上封闭且满足结合性

1) 若a\*b≠b\*a, 不妨设a\*b=a, b\*a=b, (a\*b)\*a = a\*a = b, a\*(b\*a) = a\*b = a

(a\*b)\*a≠a\*(b\*a), 与结合性矛盾, 则必有a\*b=b\*a

2) 若b\*b=a即(a\*a)\*(a\*a) = a\*(a\*a)\*a a\*b\*a= a

由(1)有a\*b=b\*a=a, a\*b\*a = a\*a = b, 或a\*b=b\*a=b, a\*b\*a = b\*a = b

与b\*b = a\*b\*a = a矛盾, 则必有b\*b=b

Problem 4

对于a∈G, 若a=a^-1, a^2 = a×a^-1 = e, |a|≦2, a为1阶或2阶元

对于|a|>2有a≠a^-1, 即G中阶大于2的元素a与b=a^-1总数成对出现

即G的阶数为偶数, G中阶大于2的元素个数为偶数,

则阶为1或2的元素个数为偶数, 1阶元只有e一个, 2阶元有奇数个(至少一个)

Problem 5

设|abc|=m, |bca| = n, (abc)^m = (bca)^n = e

(abc)^n = (abc)(abc)……(abc) = (abc)……(abc)aa^-1 = a(bca)……(bca)a^-1

= a(bca)^m a^-1 = aea^-1 = aa^-1 = e, 则有m | n

(bca)^m = (bca)(bca)……(bca) = a^-1a(bca)……(bca) = a^-1(abc)…(abc)a

= a^-1(abc)^m a = a^-1ea = a^-1a = e, 则有n | m

则m=n即|abc| = |bca|, 同理|bca| = |cab|, 则|abc| = |bca| = |cab|

Problem 6

任取|c| = |b^-1·c·b| = n, 则有c^n = (b^-1·c·b)^n = b^-1·a^n·b = b^-1·b = e

若对任意b都有b^-1 = b, 任取a^-1 = a, b^-1 = b, ab = x = x^-1, ba = y = y^-1

x = ab = a^-1·b^-1 = (ba)^-1 = y^-1 = y, 则x=y, ab=ba, 与G为非Abel群矛盾

故存在b使得b^-1≠b, 令a=b^-1, a, b非单元且ab = b^-1 b = b b^-1 = ba

Problem 7

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | 1 | -1 | i | -i |
| 1 | 1 | -1 | i | -i |
| -1 | -1 | 1 | -i | i |
| i | i | -i | -1 | 1 |
| -i | -i | i | 1 | -1 |

S≠∅, 观察表格可知乘法在S上封闭, 且同时满足结合性与交换性

对任意x∈S有1·x=x·1=x, 1为幺元, 1^-1=1, (-1)^-1 = -1, i^-1 = -i, (-i)^-1 = i

则S中所有元素都有对应的逆元, S={1, -1, i, -i}是复数上的乘法群.

Problem 8

a, b∈G, (ab)(ab)^-1 = a(b(ab)^-1)) = e, b(ab)^-1=a^-1

b^-1 (b(ab)^-1) = b^-1 a^-1 = (b^-1 b)(ab)^-1 = e(ab)^-1 = (ab)^-1

Problem 9

假设半群S有左单位元e, 对任意a∈S有e·a=a, 存在b∈S使b·a=e,

半群满足结合性, 则对任意a∈S有a·e = a·(b·a) = (a·b)·a = e·a = a

即e也是S的右单位元, (a·b)·(a·b) = a·(b·a)·b = a·e·b = a·b

令c=a·b≠e, c·c=c, 存在d∈S使d·c=e, (d·c)·c = e·c = c = d·(c·c) = d·c = e

即c=a·b=e, 矛盾, 故a·b=e, a有右逆元且等于左逆元, <S, ·>是一个群.

假设半群S有右单位元e, 对任意a∈S有a·e=a, 存在b∈S使a·b=e, 同理.

Problem 10

x^3=e, 则x的阶数n | 3, x为1阶元素或3阶元素

若x为1阶元素, x = e, 有且只有一个这样的x

若x为3阶元素, x(x^2) = e, (x^-1)^3 = (x^2)^3 = (x^3)^2 = e^2 = e

若x=x^-1, x^2=x, x^3=x^2=x=e, x为1阶元素, 矛盾, 故x≠x^-1

符合条件的x与x^-1成对出现, x为3阶元素有偶数种情形.

则G中使得x^3 = e的元素x的个数是奇数